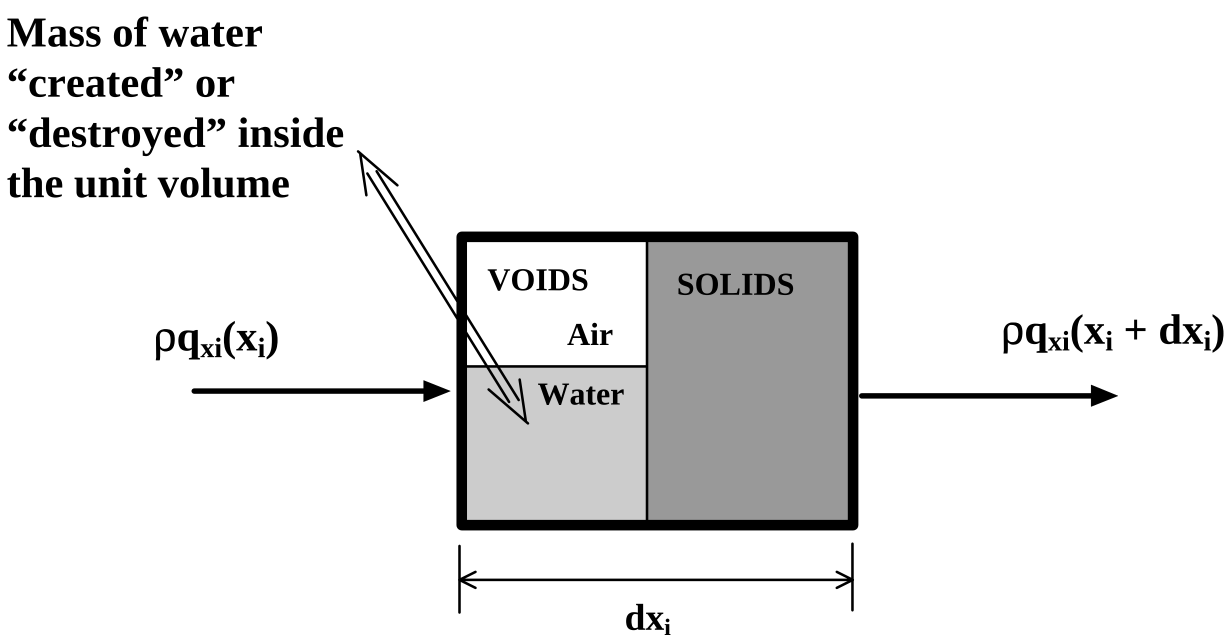
DEVELOPMENT OF WATER FLOW EQUATIONS:

Start from MASS BALANCE:

Δ Storage = Mass in - Mass out
= Σ Mass flow across unit volume
$\pm$ sources/sinks "within" the unit volume



Describe water mass balance in a porous medium using differential equations:

Assumption #1: Water is not created or destroyed (sinks and sources = 0)

Why would sinks/sources $\neq$ 0?

Mass of water in storage = ρ_w n S dx dy dz

where:
ρ_w = density of water = mass water/vol water
n = effective porosity = vol voids/vol media
S = water saturation = vol water/vol voids
θ = volumetric water content of medium = nS
dx dy dz = volume of media

$$\Delta \text{Storage} = \frac{\partial}{\partial t}\left(\rho_w n S\right) dx\, dy\, dz = \frac{\partial}{\partial t}\left(\rho_w \theta\right) dx\, dy\, dz$$

units: Mass water/time

Mass in - Mass out = (Δ flux across volume)(Area)

Mass flux of water in = $\rho_w q_x(x)$

Mass flux of water out = $\rho_w q_x(x + dx)$

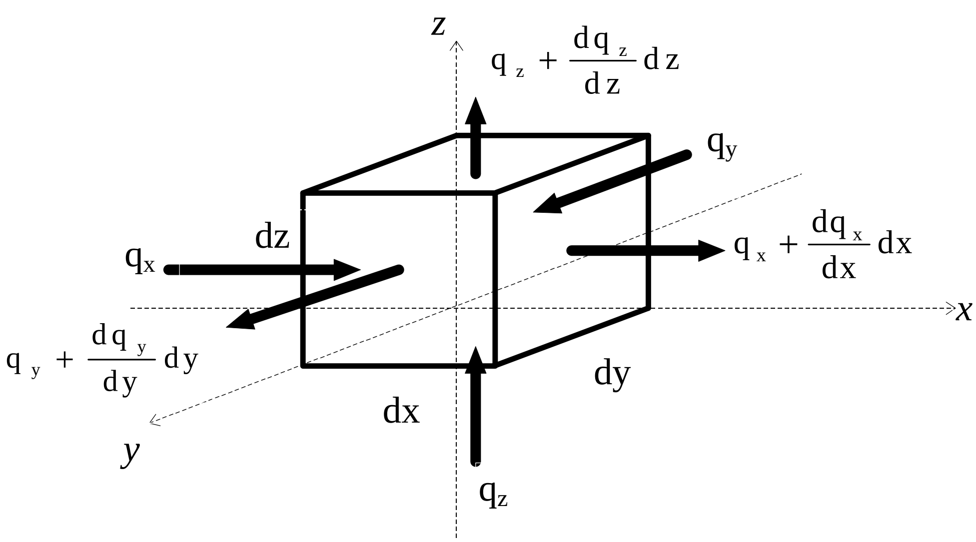
$$= \rho_w q_x + \frac{\partial \rho_w q_x}{\partial x} dx + \frac{\partial^2 \rho_w q_x}{\partial x^2} \frac{(dx)^2}{2!} + \ldots$$

$$\approx \rho_w q_x + \frac{\partial \rho_w q_x}{\partial x} dx$$

Assumption #2: Describe change in flux with truncated Taylor series expansion.

$q_x(x)$ = water flux in x direction (at x)
dy dz = Area perpendicular to x

Expand flux terms for 3 directions:



Σ Mass Flow	In	Out
Direction x	$\rho q_x \, dy \, dz$	$\rho q_x \, dy \, dz + \frac{d}{dx}(\rho q_x)\, dx \, dy \, dz$
Direction y	$\rho q_y \, dx \, dz$	$\rho q_y \, dx \, dz + \frac{d}{dy}(\rho q_y)\, dx \, dy \, dz$
Direction z	$\rho q_z \, dx \, dy$	$\rho q_z \, dx \, dy + \frac{d}{dz}(\rho q_z)\, dx \, dy \, dz$

Assemble equation components and divide by dx dy dz:

GENERAL MASS FLOW EQUATION FOR WATER IN POROUS MEDIA

$$\frac{\partial}{\partial t}(\rho_w \theta) = -\frac{\partial}{\partial x}(\rho_w q_x) - \frac{\partial}{\partial y}(\rho_w q_y) - \frac{\partial}{\partial z}(\rho_w q_z)$$

$$= -\vec{\nabla} \bullet (\rho_w \vec{q})$$

Also known as the Continuity Equation

How do we solve the equation (*usually can't measure flows*):
1 Equation, 5 unknowns (θ, ρ, q_x, q_y, q_z)

$$\frac{\partial}{\partial t}\left(\rho_w \theta\right) = -\frac{\partial}{\partial x}\left(\rho_w q_x\right) - \frac{\partial}{\partial y}\left(\rho_w q_y\right) - \frac{\partial}{\partial z}\left(\rho_w q_z\right)$$

TO SOLVE, we must either:

1. Write all variables in terms of a single variable, **hydraulic head, h,** or

2. Eliminate certain variables by simplifying assumptions/conditions.

Variable		Simplifying Condition	f(h)
ρ		Incompressible fluid ρ = constant	Specific Storage Relationship
θ	n	Nondeformable medium n = constant	
	S	Water saturation doesn't change S = constant (e.g., S = 1)	Water Capacity Curve
q_{x1}		none	Darcy/Buckingham Equation
q_{x2}		1-D flow problems $q_{x2} = 0$	
q_{x3}		1- or 2-D flow problems $q_{x3} = 0$	

OBJECTIVE: Recognize the flow equation in its various forms, and understand the assumptions/conditions required for each form.

What is HYDRAULIC HEAD? h = Energy/weight (L)
= Represented by height to which water will rise in a well

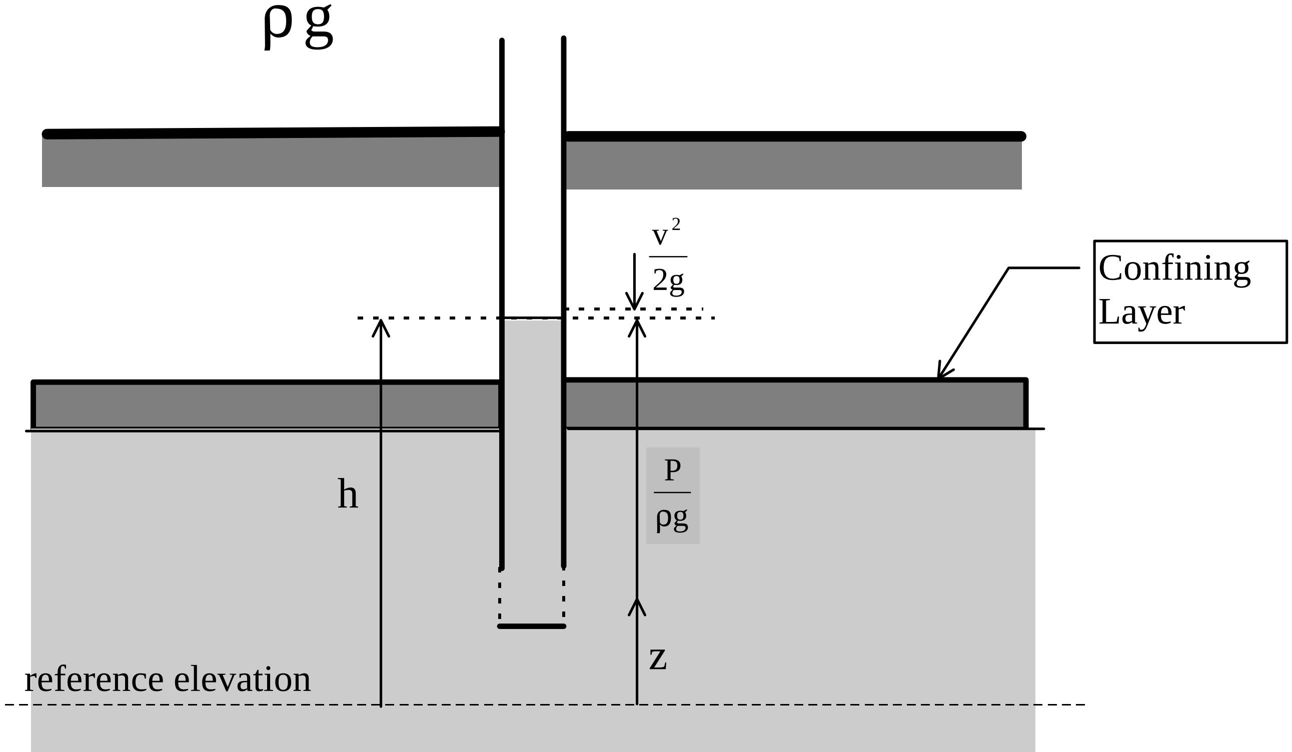
Total Energy = Potential Energy + Kinetic Energy

Type	**Energy (FL)**	**Energy/volume (FL^{-2})**	**Energy/Mass (L^2T^{-2})**	**Head (L) = Energy/Weight**
PE (Elevation)	mgz	ρgz	gz	z
PE (Pressure)	PV	P	$\frac{P}{\rho}$	$\frac{P}{\rho g}$
KE	$\frac{1}{2} m v^2$	$\frac{1}{2} \rho v^2$	$\frac{1}{2} v^2$	$\frac{v^2}{2g}$

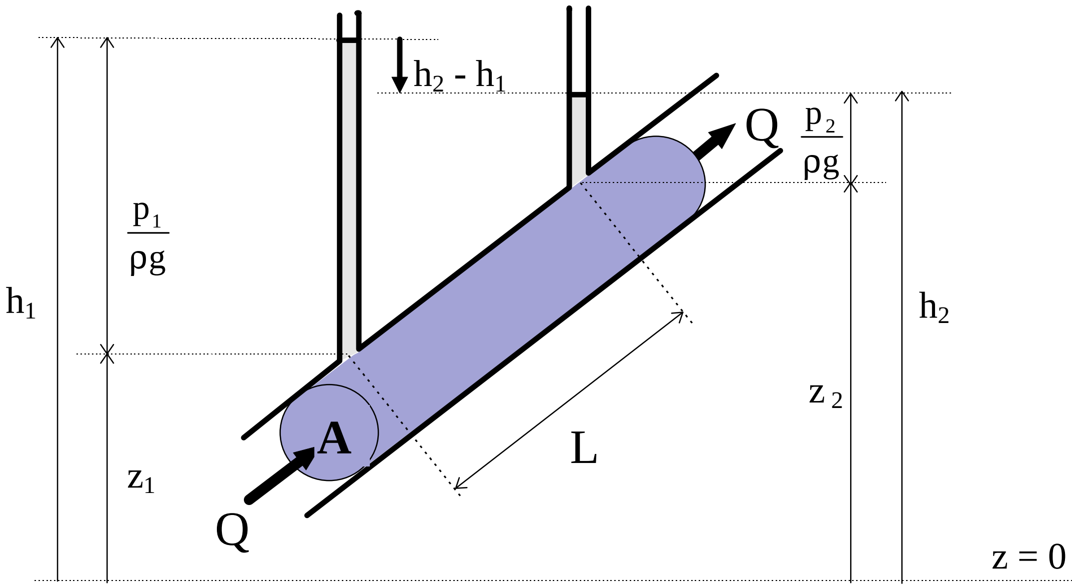
GW velocity are very small, therefore KE is neglected

Rise of water in a well represents PE.

$$h = z + \frac{P}{\rho g}$$



DARCY'S EXPERIMENT:



Darcy's Law: Flow is proportional to head gradient

$$\frac{Q}{A} = -K\frac{h_2 - h_1}{L}$$

$$q_x = -K\frac{\partial h}{\partial x}$$

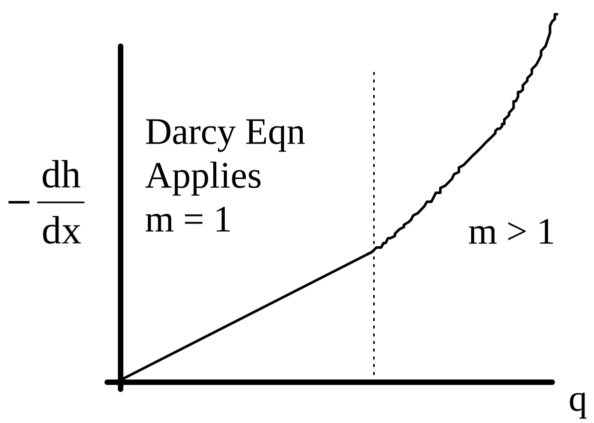
where: K = hydraulic conductivity (L/T)

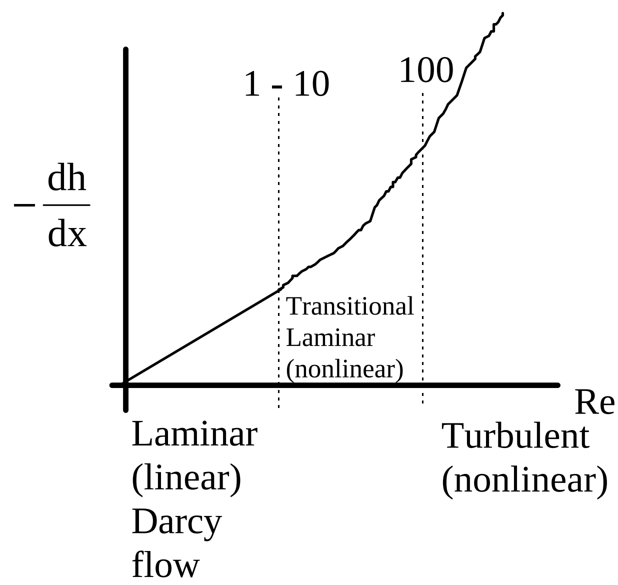
LIMITS OF DARCY'S LAW:

Darcy equation applies for LAMINAR flow.

Rewrite Darcy Equation: $\frac{dh}{dx} = -\frac{1}{K}(q)^1$

General Equation: $\frac{dh}{dx} = -\frac{1}{K}(q)^m$





Reynolds Number: $Re = \frac{q\rho d}{\mu}$

Most Groundwater Flows: Re < 1 - 10, Darcian Flow

Some Groundwater Flows: Re > 10. Nondarcian flow (e.g., Karst fissure flow)

WHAT IS SATURATED HYDRAULIC CONDUCTIVITY?

K = Measure of the ability of fluid (water) to move through porous medium
→ Units: L/T (values: see Table 3.2)
→ Increased K = increased flow

K_{sat} = f(fluid, porous medium) $K = \frac{k \rho g}{\mu}$

Where: k = Intrensic permeability (L^2)
μ = dynamic viscosity (M/LT or FT/L^2)

k = property of the porous medium
= theoretically the same for any fluid (water, air, NAPL)

why not?
1. Gas slippage
2. Water reaction with clay minerals (clay swelling)

PROPERTIES OF SATURATED HYDRAULIC CONDUCTIVITY:

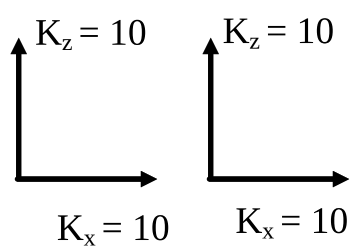
1. Uniformity: dependence on location (ln distribution)
2. Isotropy: dependence on direction (grain shape/bedding)

Uniformity: Homogeneous: K(x,y,z) = constant
Heterogeneous: K(x,y,z) ≠ constant
(K is not constant in space)

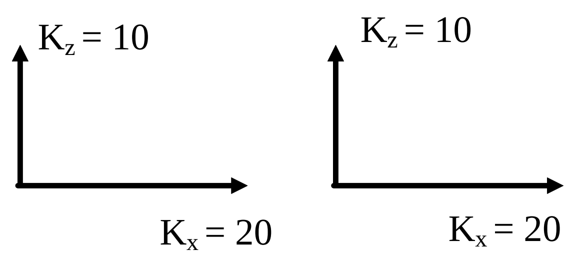
Isotropy: Isotropic: $K_x = K_y = K_z$
Anisotropic: $K_x \neq K_y \neq K_z$ (K is dependent on direction)

PROPERTIES OF SATURATED HYDRAULIC CONDUCTIVITY (continued):

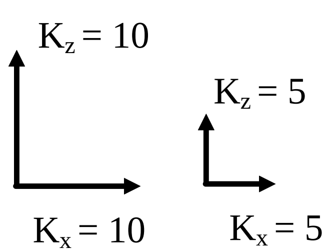
1. Homogeneous/Isotropic



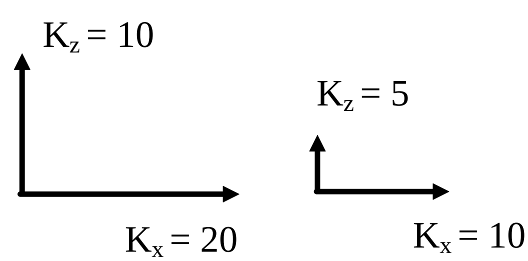
2. Homogeneous/Anisotropic



3. Heterogeneous/Isotropic



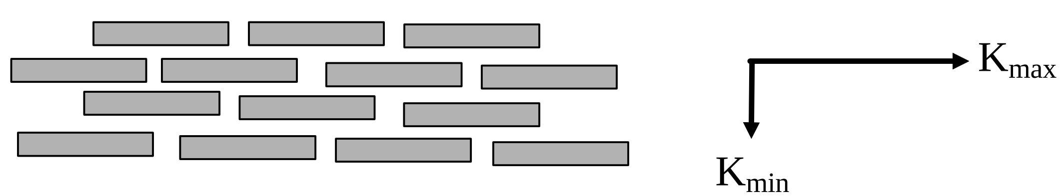
4. Heterogeneous/Anisotropic



CAUSES OF ANISOTROPY:

1. Grain Scale: Shape and orientation of solid particles:

Example: Clay grains have flattened shape:

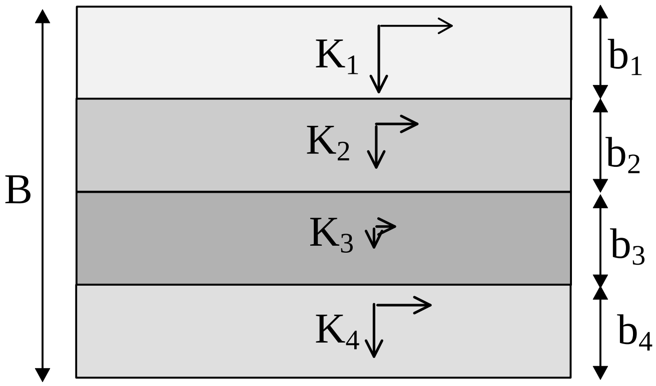


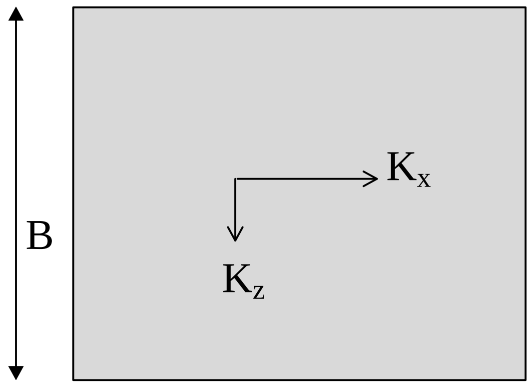
Increased conductivity parallel to platelet orientation.

2. Field scale: Bedding

Heterogeneous (bedded), isotropic media

Homogeneous, anisotropic media





Bedding is the major cause of anisotropy

Convert heterogeneous, bedded, isotropic media to homogeneous, anisotropic media $\rightarrow$ relationship between heterogeneity and isotropy:

$$\overline{K}_x = \frac{b_1K_1 + b_2K_2 + b_3K_3 + b_4K_4}{B}$$

$$\overline{K}_z = \frac{B}{\frac{b_1}{K_1} + \frac{b_2}{K_2} + \frac{b_3}{K_3} + \frac{b_4}{K_4}}$$

Insert Darcy's Law into continuity equation:

For now, assume : Incompressible Fluid
Nondeformable Medium
Water Saturated Conditions

therefore: Δ Storage = 0; and: $0 = -\vec{\nabla} \bullet \vec{q}$

$$q_x = -K \frac{dh}{dx}$$

(Darcy's law in 1-D)

WRITE DARCY'S LAW FOR 3 DIMENSIONS

1. Gradients can be in all 3 directions
2. Gradient in x_1 direction induces flow in x_2 and x_3 directions due to branching pores.

$$q_x = -K_{xx} \frac{\partial h}{\partial x} - K_{xy} \frac{\partial h}{\partial y} - K_{xz} \frac{\partial h}{\partial z}$$

$$q_y = -K_{yx} \frac{\partial h}{\partial x} - K_{yy} \frac{\partial h}{\partial y} - K_{yz} \frac{\partial h}{\partial z}$$

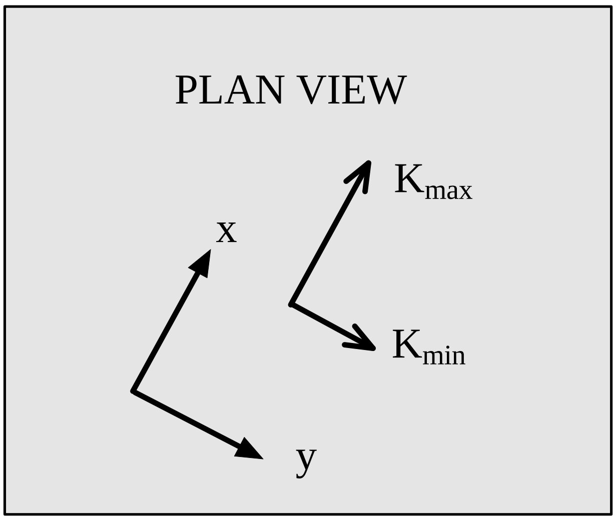
$$q_z = -K_{zx} \frac{\partial h}{\partial x} - K_{zy} \frac{\partial h}{\partial y} - K_{zz} \frac{\partial h}{\partial z}$$

Hydraulic conductivity tensor is symmetric:

$K_{xy} = K_{yx}$ $K_{xz} = K_{zx}$ $K_{yz} = K_{zy}$

If we align principle axes with direction of anisotropy - cross products go to zero

$$[K] = \begin{vmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{vmatrix}$$



Incorporate Darcy Equation into 3-D Flow Mass Balance:

$$0 = -\vec{\nabla} \bullet \vec{q}$$

1. HETEROGENOUS - ANISOTROPIC (GENERAL CASE):

$$0 = \frac{\partial}{\partial x} K_x \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K_y \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} K_z \frac{\partial h}{\partial z}$$

2. HETEROGENEOUS - ISOTROPIC

$$0 = \frac{\partial}{\partial x} K \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} K \frac{\partial h}{\partial z}$$

3. HOMOGENOUS - ANISOTROPIC:

$$0 = K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2}$$

4. HOMOGENEOUS - ISOTROPIC:

$$0 = K\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right)$$
$$= \vec{\nabla}^2 h$$

or, the Laplace equation (2nd order PDE).

What do we have?

Equation to solve for hydraulic head at any point in time/space.

What do we want?

Velocity vectors to determine direction and rate of contaminant transport.

- Use Darcy Eq'n to get *v* from *h(x,y,z)* data.
- *How?*

Groundwater Flow and Velocity Parameters:

1. Fluid flux, q

(a.k.a.: Darcy velocity or seepage velocity)

Measures: Water flow rate/Area of media

Units: L^3 *water* L^{-2} *media* T^{-1}

2. Velocity vector, v (v = q/θ)

Measures: Rate that a water "particle" moves through space ("straight line").

Units: L *media* T^{-1}

3. True groundwater velocity, v_{act} ($v_{act} = v\,\tau$)

τ = tortuosity factor ≈ 2 for saturated soil
Measures: True distance water travels over time.

Unit: L T^{-1}

Under what circumstances would we want to use each of these velocity parameters?